COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **29** (2) (2009) 215–225
DOI: 10.1007/s00493-009-2178-y

HIGHLY PARITY LINKED GRAPHS

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Received February 27, 2006
Revised February 4, 2008

A graph G is *k-linked* if G has at least $2k$ vertices, and for any $2k$ vertices $x_1,x_2,\dots,x_k,y_1,y_2,\dots,y_k$, G contains k pairwise disjoint paths $P_1,\dots,P_k$ such that P_i joins x_i and y_i for $i=1,2,\dots,k$. We say that G is *parity-k-linked* if G is k-linked and, in addition, the paths $P_1,\dots,P_k$ can be chosen such that the parities of their length are prescribed. Thomassen [22] was the first to prove the existence of a function $f(k)$ such that every $f(k)$-connected graph is parity-k-linked if the deletion of any $4k-3$ vertices leaves a nonbipartite graph.

In this paper, we will show that the above statement is still valid for $50k$-connected graphs. This is the first result that connectivity which is a linear function of k guarantees the Erdős–Pósa type result for parity-k-linked graphs.

1. Introduction

In this paper, all graphs are finite and may have loops and multiple edges. Basically, we follow the terminology of Diestel [5].

A graph G is *k-linked* if G has at least $2k$ vertices, and for any $2k$ vertices $x_1,x_2,\dots,x_k,y_1,y_2,\dots,y_k$, G contains k pairwise disjoint paths $P_1,\dots,P_k$ such that P_i joins x_i and y_i for $i=1,2,\dots,k$. We say that G is *parity-k-linked* if G is k-linked and, in addition, the paths $P_1,\dots,P_k$ can be chosen such that the parities of their length are prescribed. The study of k-linked

Mathematics Subject Classification (2000): 05C40, 05C83

* Research partly supported by the Japan Society for the Promotion of Science for Young Scientists, by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research and by Inoue Research Award for Young Scientists.

graphs has long history. Jung [8] and Larman and Mani [12], independently, proved the existence of a function $f(k)$ such that every $f(k)$-connected graph is k-linked. Bollobás and Thomason [2] was the first to prove that linear connectivity is enough, i.e., they proved that every $22k$-connected graph is k-linked. Very recently, Kawarabayashi, Kostochka and Yu [9] proved that every $12k$-connected graph is k-linked, and finally, Thomas and Wollan [20] proved that every $10k$-connected graph is k-linked. Actually, they proved the following stronger statement.

Theorem 1.1 **([20]).** *Every $2k$-connected graph with at least $5k|V(G)|$ edges is k-linked.*

In this paper, we are interested in parity-k-linked graphs. It is easy to see that no matter how large the connectivity is, a bipartite graph may not be parity-k-linked. The natural analogue of Theorem 1.1 is:

Suppose G is $f(k)$-connected, where $f(k)$ is some function of k. Then there is a function $g(k)$ such that either G is parity-k-linked or G has a vertex set X of order at most $g(k)$ such that $G-X$ is bipartite.

In [22], Thomassen proved that for any integer k, a $2^{3^{27k}}$-connected graph is parity-k-linked if the deletion of any set of less than $4k-3$ vertices leaves a nonbipartite graph.

The connectivity bound "$4k-3$" is best possible in a sense. Consider a large complete bipartite graph with making $2k-1$ vertices in one of the bipartite classes the complete graph, and with making $2k$ vertices the complete graph minus a perfect matching in the other bipartite class. This graph shows that it is not parity-k-linked if there is a set X of $4k-4$ vertices in a graph G such that $G-X$ is bipartite.

On the other hand, the connectivity seems to be far from best possible. In fact, Thomassen [22] conjectured that linear connectivity will do the same. The purpose of this paper is to prove his conjecture. Our main theorem is the following.

Theorem 1.2. *Suppose G is $50k$-connected. Then either G is parity-k-linked or G has a set X of less than $4k-3$ vertices such that $G-X$ is bipartite.*

As we pointed out, the bound "$4k-3$" is best possible. Also, the connectivity is essentially best possible since linear connectivity is necessary for a given graph to be k-linked.

These results were partially motivated by the study of the Erdős–Pósa property for odd cycles in special classes of graphs.

A family $\mathcal{F}$ of graphs is said to have the *Erdős–Pósa property*, if for every integer k there is an integer $f(k,\mathcal{F})$ such that every graph G contains either k vertex-disjoint subgraphs each isomorphic to a graph in $\mathcal{F}$ or a set C of at most $f(k,\mathcal{F})$ vertices such that $G-C$ has no subgraph isomorphic to a graph in $\mathcal{F}$. The term *Erdős–Pósa property* arose because in [6], Erdős and Pósa proved that the family of cycles has this property.

On the other hand, for odd cycles, the situation is different.

The family of odd cycles does not have the Erdős–Pósa property, as we now show. For a graph G an *odd cycle cover* is a set of vertices $C \subseteq V(G)$ such that $G-C$ is bipartite.

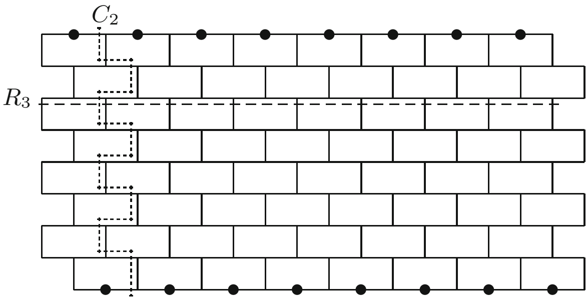

Figure 1. An elementary wall of height 8

An elementary wall of height eight is depicted in Figure 1. An *elementary wall* of height h for $h \geq 3$ is similar. It consists of h levels each containing h bricks, where a brick is a cycle of length six. A *wall* of height h is obtained from an elementary wall of height h by subdividing some of the edges, i.e., replacing the edges with internally vertex disjoint paths with the same end-points (see Figure 2).

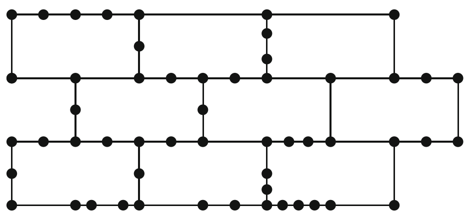

Figure 2. A wall of height 3

An *Escher wall of height* h consists of a wall of height h: W, and h vertex disjoint paths $P_1,\dots,P_h$ such that:

(i) Each P_i has both endpoints on W but is otherwise disjoint from W.
(ii) One endpoint of P_i is in the ith brick of the top row of bricks of W, the other is in the $(h+1-i)$th brick of the bottom row of W. Furthermore, both of these vertices are in only one brick of W.
(iii) W is bipartite but for each i, $W \cup P_i$ contains an odd cycle.

See Figure 3 for an example.

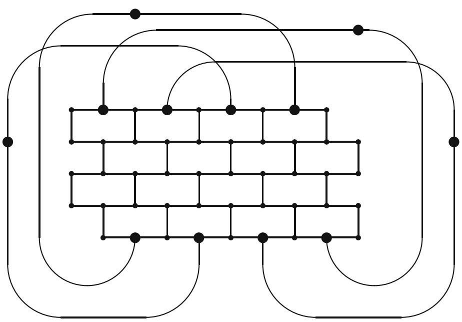

Figure 3. An Escher Wall of height 4

We remark that, as pointed out by Lovász and Schrijver (see [19]), an Escher wall W of height h contains neither 2 vertex disjoint odd cycles nor an odd cycle cover with fewer than h vertices (the first fact follows from the fact that for any Escher wall $W, P_1,\dots,P_k$, the planar embedding of W can be extended to an embedding of the Escher wall in the projective plane so that every odd cycle is non-null homotopic by routing the P_i through a cross-cap; the fact that there is no odd cycle cover of size $h-1$ follows from the fact that if X is a set of $h-1$ vertices then X fails to intersect some path along the top of a level of bricks of W and then, as is easily verified, there is some i such that P_i is disjoint from X and both endpoints of P_i are connected to this row in $W-X$. Hence the two endpoints of this P_i are connected by a path Q in $W-X$ and we have that P_i+Q is an odd cycle in $G-X$). This shows that the Erdős–Pósa property does not hold for odd cycles (in fact, it holds for the cycles of length $p \ mod \ m$ if and only if p is congruent to 0 $mod \ m$ see [4] and [21]).

While the Erdős–Pósa property does not hold for odd cycles in general, Reed [18] proved that the Erdős–Pósa property holds for odd cycles in planar graphs. This result was extended to an orientable fixed surface in [10]. Note that the Erdős–Pósa property does not hold for odd cycles in nonorientable surfaces, even for projective planar graphs as the above example shows.

But an Escher wall cannot be 4-connected, so one can hope that if a graph is highly connected compared to k, then the Erdős–Pósa property holds for odd cycles. Motivated by this, Thomassen [22] was the first to prove that there exists a function $f(k)$ such that every $f(k)$-connected graph G has either k disjoint odd cycles or a vertex set X of order at most $2k-2$ such that G–X is bipartite. Hence, he showed that the Erdős–Pósa property holds for odd cycles in highly connected graphs. Soon after that, Rautenbach and Reed [17] proved that the connectivity $f(k) \leq 576k$ will do the same. The bound "$2k-2$" is best possible in a sense since a large complete bipartite graph with the edges of a complete graph of $2k-1$ vertices added in one partite set does not contain k disjoint odd cycles.

Our proof implies the connectivity $576k$ in [17] can be improved to $24k$. See the end of the next section.

2. Proof of the main result

To prove our main result, we may assume that G has an odd cycle cover of order less than $4k-3$. We need to show that for any set X of $2k$ specified vertices $\{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$, we can find k disjoint paths $T_1, \ldots, T_k$ such that T_i joins x_i and y_i, and T_i has a prescribed parity for $i = 1, \ldots, k$. First, we claim the following.

(1) *We may assume that $G - X$ has an odd cycle cover of order at most $4k-2$.*

Suppose the contrary and for some l with $2k-2 \leq l \leq 4k-3$, let $U = \{u_1, \ldots, u_l\}$ be a minimum odd cycle cover. Note that $l \geq 2k-2$ since G has no odd cycle cover of order less than $4k-3$. Let A, B be partite sets of $G - X - U$. Clearly $|A|, |B| \geq 44k$ and any vertex in U has at least $44k$ neighbors in $G - X - U$, since G is $50k$-connected. We can partition U in two sets U_A and U_B such that every vertex of U_A has at least $20k$ neighbors in A and every vertex of U_B has at least $20k$ neighbors in B. We may assume that $U_A = \{u_1, \ldots, u_r\}$ and $U_B = \{u_{r+1}, \ldots, u_l\}$ for some $0 \leq r \leq l$.

We claim that there are a matching M_A in the graph $G[U_A \cup B]$ that covers all of U_A and a matching M_B in $G[U_B \cup A]$ that covers all of U_B. Otherwise, there is no matching in $G[U_A \cup B]$, say, that covers all of U_A. Using a small extension of Tutte's matching theorem (see for instance [13]), this implies the existence of a set Z of vertices such that the graph $G[U_A \cup B] - Z$ has at least $|Z|+1$ odd components that are contained in U_A. Let the set Z' contain exactly one vertex from each of these components. Then the graph $H = G - X - ((U - Z') \cup Z)$ is bipartite with bipartition $((A \cap V(H)), ((B \cup Z') \cap V(H)))$. Hence the set $(U - Z') \cup Z$ is an odd cycle cover of order $|U| - |Z'| + |Z| < |U|$ which contradicts the choice of U.

Now we shall find the desired k disjoint paths in G as follows.

We choose different vertices $a_1, \dots, a_r \in A$ and $b_{r+1}, \dots, b_l \in B$ such that for $1 \le i \le r$ the vertex a_i is a neighbor of u_i, for $r+1 \le i \le l$ the vertex b_i is a neighbor of u_i, and none of these vertices is incident with an edge in $M_A \cup M_B$. Such a choice is possible, since the vertices in U_A and U_B have at least $20k$ neighbors in A and B, respectively.

We say that P is a *parity breaking path* if $G - X - U$ together with P has an odd cycle of G.

For every vertex u_i in U_A that is matched to a vertex b in B, we can find a parity breaking path b, u_i, a_i.

For every vertex u_i in U_A that is matched to a vertex u_j in U_A, we can find a parity breaking path u_j, u_i and their two neighbors a_i and a_j in A. Similarly, we can construct parity breaking paths for U_b.

We now have at least $\lceil \frac{l}{2} \rceil \ge k - 1$ disjoint parity breaking paths, and if there are exactly $k - 1$ disjoint parity breaking paths, then this implies that $U \cup X$ is an odd cycle cover of order either $4k - 2$ or $4k - 3$, and in the first case, $M(A) \cup M(B)$ uses only vertices in U, while in the second case, only one edge in $M(A) \cup M(B)$ uses a vertex in $G - X - U$.

Take a matching from X to $G - X - U - V(M(A) \cup M(B)) - \{a_1, \dots, a_r, b_{r+1}, \dots, b_l\}$. This is possible since every vertex of X has degree at least $50k$. Since $G - U - X$ is still $44k$-connected, by Theorem 1.1, $G - U - X$ is $4k$-linked. For each j, we will either link x_j to y_j directly or link x_j and y_j to the two endvertices of one of parity breaking paths in $G - U - X$, since exactly one of these options will give us a path of the prescribed parity. Since $G - U - X$ is $4k$-linked, if there are at least k parity breaking paths, we can find the desired k disjoint paths.

It remains to consider the case in which there are exactly $k - 1$ parity breaking paths, in which case, $U \cup X$ is an odd cycle cover of size either $4k - 2$ or $4k - 3$ for G. In order to finish the proof, we only need to consider the case where either $l = 2k - 3$ or $l = 2k - 2$. In the first case, we may assume that U_A is odd and U_B is even. We may also assume that u_1 is the unique vertex in U_A which is matched to a vertex b in B by the matching M_A. Suppose that a vertex in X, say x_1, sees vertices in both sides of the bipartition of $G - X - U - \{b\}$. It follows that we can find a matching $x_1 z_1, y_1 w_1$ for two vertices w_1, z_1 of $G - X - U$ such that a path from z_1 to w_1 in $G - X - U$ together with $x_1 z_1 y_1 w_1$ gives a desired parity for the pair (x_1, y_1). Now, we can insist that none of the a_i or b_i are in $\{z_q, w_1\}$ as we have over $20k$ choices for each such a vertex. Thus, we can proceed as above as we need only $k-1$ parity breaking paths. Similarly, if x_1 and y_1 see different sides in $G-X-U-\{b\}$, or x_1 and y_1 see the same side of $G-X-U-\{b\}$ and $x_1y_1 \notin E(G)$, then either $(X \cup U)-\{x_1,y_1\}$ or $((X \cup U)-\{x_1,y_1\}) \cup \{b\}$ would be an odd cycle cover of smaller order, a contradiction. Note that every x_i sees only one side of $G-X-U-\{b\}$ and every y_i sees only one side of $G-X-U-\{b\}$. Thus x_1 and y_1 see the same side of $G-X-U-\{b\}$ and $x_1y_1 \in E(G)$. But then the pair (x_1,y_1) does not need any parity breaking path, and hence we only need $k-1$ parity breaking paths. Hence $G-X$ has an odd cycle cover of order at most $4k-2$. This completes the proof of (1).

Let $G' = G - X$. Then G' is $48k$-connected. As observed by Erdős, if we choose a spanning bipartite graph H of G' with maximum number of edges, then the minimum degree of H is at least $24k$, and hence H has at least $12k|V(H)|$.

(2) *H has a $2k$-linked subgraph K.*

To prove (2), we shall need the following result. This result was originally proved in [1], which was used to prove that there is a function $f(k)$ such that every $16k$-connected graph with at least $f(k)$ vertices has a K_k-minor. But for the completeness, we shall give a proof here since what we need in this paper is a "triangle-free" version of the theorem in [1].

(2.1) *Let G be a triangle-free graph and k an integer such that*

(a) $|V(G)| \geq 12k$ *and*
(b) $|E(G)| \geq 6k|V(G)| - 12k^2$.

Then G contains a $2k$-connected subgraph H with at least $5k|V(H)|$ edges.

Suppose that (2.1) is not true. Let G be a triangle-free graph with n vertices and m edges, and let k be an integer such that (a) and (b) are satisfied. Suppose, moreover, that

(c) G contains no $2k$-connected subgraph H with at least $5k|V(H)|$ edges, and
(d) n is minimal subject to (a), (b) and (c).

Claim 1. $|V(G)| \geq 12k+1$.

By Turan's theorem (see [5]), a triangle-free graph with n vertices has at most $n^2/4$ edges. Hence, if G is a triangle-free graph on n vertices with at least $6kn - 12k^2$ edges, then $6kn - 12k^2 \leq n^2/4$. Thus either $n \leq 12k - \sqrt{96k^2} < 3k$ or $n \geq 12k + \sqrt{96k^2} > 12k$. Hence the result holds.

Claim 2. *The minimum degree of G is more than $6k$.*

Suppose that G has a vertex v with degree at most $6k$, and let G' be the graph obtained from G by deleting v. By (c), G' does not contain a $2k$-connected subgraph H with at least $5k|V(H)|$ edges. Claim 1 implies that $|V(G')| = n-1 \geq 12k$. Finally, $|E(G')| \geq m-6k \geq 6k|V(G')| - 12k^2$. Since $|V(G')| < n$, this contradicts (d) and the claim follows.

Claim 3. $m \geq 5kn$.

Suppose $m < 5kn$. Then $6kn - 12k^2 < 5kn$. Hence $n \leq 12k$, a contradiction to Claim 1.

By Claim 3 and (c), G is not $2k$-connected. This implies that G has a separation (A_1, A_2) such that $A_1 \setminus A_2 \neq \emptyset \neq A_2 \setminus A_1$ and $|A_1 \cap A_2| \leq 2k-1$. By Claim 2, $|A_i| \geq 6k+1$. For $i \in \{1,2\}$, let G_i be a subgraph of G with vertex set A_i such that $G = G_1 \cup G_2$ and $E(G_1 \cap G_2) = \emptyset$. Suppose that $|E(G_i)| < 6k|V(G_i)| - 12k^2$ for $i = 1,2$. Then

$$6kn - 12k^2 \leq m = |E(G_1)| + |E(G_2)| < 6k(n + |A_1 \cap A_2|) - 24k^2 \leq 6kn - 12k^2,$$

a contradiction. Hence, we may assume that $|E(G_1)| \geq 6k|V(G_1)| - 12k^2$. Now $|V(G_1)| \geq 6k$ by Claim 2. In fact, by the proof of Claim 1, $|V(G_1)| \geq 12k$. Thus, as in Claim 3, $|E(G_1)| \geq 5k|V(G_1)|$. Furthermore, any $2k$-connected subgraph H of G_1 with at least $5k|H|$ edges is also a subgraph of G. So G_1 contradicts (d) and (2.1) is proved.

Combine (2.1) with Theorem 1.1, (2) holds.

Let K be a $2k$-linked bipartite subgraph of G'. We say that P is a *parity breaking path* for K if P has only endpoints in K and K together with P has an odd cycle of G. This parity breaking path may be just a single edge. We shall use the following recent result in [7,3].

(3) *For any set S of vertices of a graph G, either*

1. *there are k disjoint odd S paths, i.e., k disjoint paths each of which has an odd number of edges and both its endpoints in S, or*
2. *there is a vertex set X of order at most $2k-2$ such that $G - X$ contains no such paths.*

One of the partite set of K has at least $2k$ vertices.

(4) *There are at least $2k$ disjoint parity breaking paths for K.*

Take such a partite set S of K with at least $4k$ vertices. We shall apply (3) to G' and S. Note that $|S| \geq 4k$ since K is $4k$-connected by (2.1). If there are at least k vertex-disjoint odd S paths in G', we can clearly find at least $2k$ vertex-disjoint parity breaking paths for K since K is a bipartite subgraph. Otherwise, there is a vertex set R of order at most $4k-2$ such that $G'-R$ has no such paths. Since $|R| \le 4k-2$ and G' is $48k$-connected, it follows that $G'-R$ is 2-connected. If there is an odd cycle C in $G'-R$, then we can take two disjoint paths from C to S, and this would give an odd S path, a contradiction. This implies that $G'-R$ is bipartite. But then there is a vertex set R of order at most $4k-2$ such that $G'-R$ is bipartite. This contradicts (1).

We shall construct the desired k disjoint paths by using K and $2k$ parity breaking paths $P_1,\dots,P_{2k}$. Let $P = \bigcup_{l=1}^{2k} E(P_l)$. Since G is $50k$-connected, there are $2k$ disjoint paths from X to K. We take a set of $2k$ disjoint paths $\mathcal{W}=\{W_1,\dots,W_{2k}\}$ joining X with K such that $|\{P_1,\dots,P_{2k}\}\cap\mathcal{W}|$ is as small as possible, and subject to that, $\bigcup_{l=1}^{2k}(E(W_l)\cap P)$ is as large as possible. We assume W_i joins x_i with K for $i=1,2,\dots,k$ and W_{i+k} joining y_i and K for $i=1,2,\dots,k$. Let x_i' be the other endvertex of the path W_i. Similarly, let y_i' be the other endvertex of the path W_{i+k}. Note that both x_i' and y_i' are in K.

Let J be indices j such that precisely one path in $\mathcal{W}$ intersects the path P_j. Let $\mathcal{W}'$ be the paths W_i such that W_i intersects a path P_j with $j \in J$. Suppose that precisely one path, say $W \in \mathcal{W}'$, intersects a path P_i. Then our choice implies that W follows the path P_i and ends at one of the endvertex. Let us observe that W can control the parity since P_i is a parity breaking path. If at least two paths of $\mathcal{W}$ intersect a path P_i, then let W and W' be the paths that intersect P_i as close as possible (on P_i) to one endvertex of P_i and the other endvertex of P_i, respectively. Our choice implies that both W and W' follow the path P_i and end at the endvertices of P_i. Note that both x_i' and y_i' are in K. Therefore, there are at least $k-\frac{|\mathcal{W}'|}{2}$ parity disjoint paths disjoint from W.

If $W_i, W_{i+k} \notin \mathcal{W}'$, then we need to connect x_i' and y_i' through one of the parity breaking path with no intersection with $\mathcal{W}$. If at least one of W_i, W_{i+k} is in $\mathcal{W}'$, then we only need a path from x_i to y_i in K since we can control the parity of either W_i or W_{i+k}. Since K is $2k$-linked and there are at least $k-\frac{|\mathcal{W}'|}{2}$ parity breaking paths disjoint from W, it is possible to find the desired k disjoint paths. Let us observe that if there is a path $W \in \mathcal{W}$ such that W only hits an endpoint of P_i, and does not hit any other path, then we regard W to hit P_i. Moreover, if this path W is in $\mathcal{W}'$, then we cannot control the parity of W. But in this case, we think of this parity path P_i to adjust the parity of W, if necessary. This will not harm our argument, since K is a $2k$-linked subgraph. This completes the proof. ∎

Let us observe that if there is a k-linked bipartite subgraph with k disjoint parity breaking paths, then we get k disjoint odd cycles. The argument in (2) implies that if the minimum degree of a given graph is at least $24k$, then it has a k-linked bipartite subgraph K with partite set (A,B) with $|A| \geq |B|$. The argument in (4) implies that if there are no k disjoint parity breaking paths for K, then there is a vertex set R of order at most $2k-2$ such that $G-R$ has no odd A paths. But $G-R$ is 2-connected, so if there is an odd cycle C in $G-R$, then there are two disjoint paths from C to A. But clearly we can take an odd A path, contrary to (3). This implies that $G-R$ is bipartite. Hence the arguments in (2) and (4) imply the following.

Theorem 2.1. *Every $24k$-connected graph has either k disjoint odd cycles or a vertex set X of order at most $2k-2$ such that $G-X$ is bipartite.*

Theorem 2.1 improves the connectivity $576k$ in [17] to $24k$.

Acknowledgement

We would like to thank the referees for suggesting improvements for both proofs and presentations.

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